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Calculus (Tutorial # 5)

Differentiation in $\mathbb{R}$

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|f(x) - f(y)| \leq |x - y|^{\alpha}, \quad \text{for some } \alpha \in \mathbb{R}. \tag{1}$$

   (a) If $0 < \alpha \leq 1$, then show that f is continuous on $\mathbb{R}$. Is it true that such an f uniformly continuous on $\mathbb{R}$? Also give an example of a non-constant function f which satisfies (1) with $\alpha = \frac{1}{2}$.

   (b) If $\alpha > 1$, then show that f is differentiable on $\mathbb{R}$. What will be your conclusion if $\alpha = 1$?

   (c) Find all the functions $f : \mathbb{R} \to \mathbb{R}$ which satisfies (1) with $\alpha > 1$.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Find $f'(x)$ for values of $x \neq 0$.

   (b) Verify that f is also differentiable at $x = 0$ and evaluate $f'(0)$.

   (c) Verify that f' is not continuous at $x = 0$.

   (d) Check the uniform continuity of f.

3. If f is differentiable at a, then $|f|$ is also differentiable at a if $f(a) \neq 0$. Show that this does not hold true if $f(a) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

   Then show that $f \in C^{\infty}(\mathbb{R})$ and $f^{(k)}(0) = 0$, for each $k \in \mathbb{N}$.

5. Let f have continuous first and second derivatives in (a, b). Then prove that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

$$\text{and} \qquad f^{(2)}(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

6. Let $f : I := (a, b) \to \mathbb{R}$ be continuously differentiable with $f'(x) \neq 0$, for all $x \in I$. Then

   (a) f is strictly monotone.

   (b) $f(I) = J$ is an interval.

   (c) $g := f^{-1}$ is continuously differentiable function on the interval $J := f(I)$ and we have
   $$g'(f(x)) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}, \text{ for all } x = g(y) \in (a, b)$$

7. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function. If $\lim_{x \to \infty} f'(x) = l$ then show that $\lim_{x \to \infty} \frac{f(x)}{x} = l$.

8. True or False? Justify your answer.

   (a) Suppose $f : (4, 6) \to \mathbb{R}$ is a bounded differentiable function and $f'(c) = 0$ for some $c \in (4, 6)$. Then c is a local maxima or minima for f.

   (b) If f is a real valued function on $S = \{x \in \mathbb{R} : |x| > 5\}$ such that $f'(x) = 0$ for all x, then f is constant on S.

   (c) The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{\sin(|x|)}$ is differentiable at 0.

   (d) If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, and $f'(c) > 0$, then f is increasing in $(c - \delta, c + \delta)$ for some $\delta > 0$.

   (e) Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function, such that $f(c)$ is the maximum value of f on $[a, b]$ for some $c \in (a, b)$. Then $f'(c) = 0$.

   (f) If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $f''(3) > 0$, then f has local minimum at 3.

   (g) Suppose $f : [a, b] \to \mathbb{R}$ is a function which is differentiable on (a, b). If $f(a) = f(b) = 5$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

   (h) If $f : [a, b] \to \mathbb{R}$ is a function, which is differentiable on (a, b), then it is uniformly continuous.

   (i) $f(x) = \frac{1}{x}$ is convex on $(0, \infty)$.

   (j) $f(x) = x \log x$ is convex on $(0, \infty)$

   (k) $f(x) = x^\alpha$ is concave on $(0, \infty)$ for $0 < \alpha < 1$.

   (l) $\frac{x}{1+x} \leq \log(1 + x) \leq x$ for $x > 0$.

9. Use derivatives to prove the Bernoulli's inequality:
   $$(1 + x)^n > 1 + nx \quad (x > -1 \text{ and } x \neq 0)$$
   (Hint: consider $f(x) = (1 + x)^n - 1 - nx$, and show f is strictly increasing by showing $f'(x) > 0$ for $x > -1$.)

10. Using derivatives, prove the following:

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}, \text{ for all } x \in \mathbb{R}.$$

(Hint: Consider $f(x) = \frac{x}{1+x}$; show that it is increasing; then use $|x+y| \leq |x| + |y|$ to estimate $f(|x+y|)$.)

11. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Assume that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$. Then show that $f(x) \leq g(x)$ for all $x \geq 0$. Using this result prove the following:

    (a) $\sin x \leq x$ for all $x \geq 0$.

    (b) $1 - \frac{x^2}{2} \leq \cos x$ for all $x \geq 0$.

    (c) $1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ for all $x \geq 0$.

12. Let $n \geq 2$, $r > 0$. Let $f^{(n)}$ be continuous on $[a-r, a+r]$. Assume that $f^{(k)}(a) = 0$ for $1 \leq k \leq n-1$, but $f^{(n)}(a) \neq 0$. If n is even, then a is a local extremum. It is a minimum if $f^{(n)}(a) > 0$ and a local maximum if $f^{(n)}(a) < 0$. If n is odd, then a is a point of inflection.

13. Show that $\cos x = x^3 + x^2 + 4x$ has exactly one root in $[0, \pi/2]$.

14. Prove that the equation $x^3 - 3x^2 + b = 0$ has atmost one root in the interval $[0, 1]$. Determine the values of b for which it has only one root in $[0, 1]$.

15. Find the maximum and minimum values of the function $f(x) = 2x^3 - 3x^2 - 12x + 8$ on each of the following intervals.

    (a) $[-2.5, 4]$

    (b) $[-2, 3]$

    (c) $[-2.25, 3.75]$.

16. Find the Taylor's series for $\cos x$ about $\pi/3$ i.e. in power of $(x - \pi/3)$.

17. Find the Taylor polynomial of degree 4 for $f(x) = \tan^{-1} x$ and estimate the remainder in

$$\tan^{-1} x = x - \frac{x^3}{3} + (\text{remainder}), \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

18. Give an example of convex functions f and g, for which the composition fog is not convex.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and convex. If f is bounded, then f is constant.

20. Over which intervals in $\mathbb{R}$, the following functions are convex?

    (a) $f(x) = 5x^4 - 3x^3 + x^2 - 1$
    (b) $f(x) = \dfrac{x+1}{x-1}$
    (c) $f(x) = \sqrt{x}$
    (d) $f(x) = \dfrac{1}{x}$
    (e) $f(x) = \sqrt{1-x^2}$
    (f) $f(x) = e^{-x^2}$.

21. Find the local extreme values of $f(x) = 2x^3 - 3x^2 + 12x$. On which intervals is f convex? concave? Sketch a graph of f based on this information.

22. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Show that the following are equivalent.

    (a) f is convex.
    (b) f' is an increasing function on (a, b).
    (c) $f(y) \geq f(x) + f'(x)(y - x)$ for all $x, y \in (a, b)$.

23. Use L'Hospital evaluate the following limits.

    (a) $\lim_{y\to\infty} e^{-1/y}$.
    (b) $\lim_{y\to\infty} y^2 e^{-1/y}$.
    (c) $\lim_{y\to\infty} \dfrac{e^{-y}}{1-e^{-1/y}}$.
    (d) $\lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x}\right)$.
    (e) $\lim_{x\to 0} \dfrac{\sin x - x\cos x}{x^3}$.
    (f) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$.
    (g) $\lim_{x\to 0} \dfrac{x}{e^x - 1}$.
    (h) $\lim_{x\to 0} \dfrac{\sin x}{x}$.

24. Using the Leibniz rule evaluate the following:

    (a) 5th order derivative of $f(x) = x^2 \sin 2x$ at $x = 0$.
    (b) nth derivative of $f(x) = x \log x$.
    (c) nth derivative of $f(x) = \tan^{-1} x$ at $x = 0$.
    (d) 4th derivative of $f(x) = e^x \sin x$.
    (e) 3rd derivative of $f(x) = e^x \log x$.